Mirror Symmetry: Lecture 1

Ref.: P. Seidel, "Fukaya categories and Picard-Lefschetz theory"

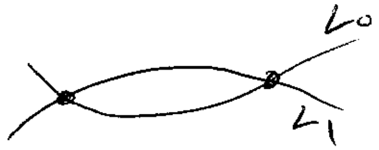
(M, ω) symplectic, exact $\omega = d\theta$

$\cup$

L exact Lagrangian submanifolds, $\omega|_L = 0$

$\theta|_L$ exact 1-form

Lagrangian Floer ~~complex~~ theory:



$L_0 \pitchfork L_1$

Lag. Floer complex: $CF(L_0, L_1) = \bigoplus_{p \in L_0 \cap L_1} K\langle p \rangle$

$\partial : CF(L_0, L_1) \circlearrowleft$ counts pseudo-holomorphic strips

$$\frac{\partial u}{\partial s} + J(u) \frac{\partial u}{\partial t} = 0$$

$\partial^2 = 0$ (always satisfied if Lagrangians are exact)

$$HF^*(L_0, L_1) = H^*(CF, \partial)$$

invt under Ham. isotopies

Fukaya category: A_∞-category

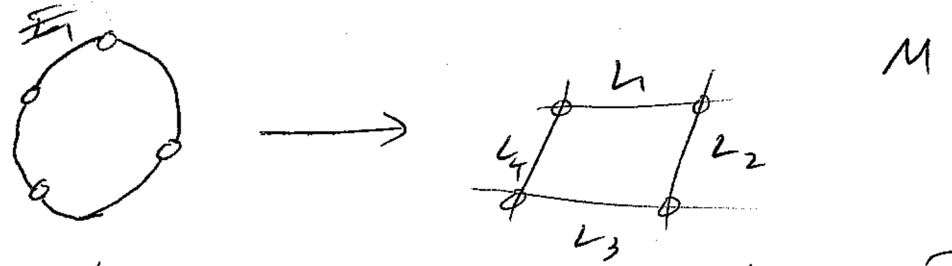
→ objects: closed, exact, Lag. sub. mflds + (extra data, spin, ..)

→ $\hom(L_0, L_1) = CF(L_0, L_1)$

$$\mu^k : \hom(L_0, L_1) \otimes \hom(L_1, L_2) \otimes \cdots \otimes \hom(L_{k-1}, L_k) \to \hom(L_0, L_k)[2-k]$$

μ^1 = differential, μ^2 = product/composition assoc. upto homotopy

Roughly, μ^k counts (perturbed) pseudo-holomorphic maps



Goal: understand $\mathcal{F}(M)$? $\longrightarrow$ $\left(\begin{array}{l} Tw\ \mathcal{F} \\ mod\ \mathcal{F} \\ D^\pi \mathcal{F} \end{array} \right)$ triangulated.

- find a few objects
- compute their Floer theory
- see what this gets about other objects.

Triangulated categories:

$$\begin{tikzcd} L_0 \arrow[rr, "p"] & & L_1 \arrow[dl, "q"] \\ & L_2 \arrow[ul, "r"', "\deg 1"] & \end{tikzcd}$$

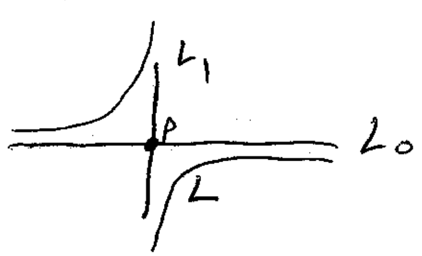
think of: mapping cones

$\forall\, T$, $\exists$ long exact sequences

$$hom^i(T, L_0) \xrightarrow{p^\circ} hom^i(T, L_1) \xrightarrow{q^\circ} hom^i(T, L_2)$$

$$\xrightarrow{r^\circ} hom^{i+1}(T, L_0) \to \cdots$$

$L_0, L_1 \in \mathcal{F}(M)$: $L_0 \pitchfork L_1$ at a single point p



then $\longrightarrow$

$$\begin{tikzcd} L_0 \arrow[rr, "p"] & & L_1 \arrow[dl] \\ & L \arrow[ul] & \end{tikzcd}$$

Q: Do $L_1, \ldots, L_r$ generate $\mathcal{F}(M)$?

Conj: (Arnold) closed

$L \subset T^*N \overset{?}{\Longrightarrow} L \sim$ zero section?

closed, exact Lag. Ham. isotopic

Thm: (Fukaya-Seidel-Smith, Nadler-Zaslow, Abouzaid)

assumptions $\Longrightarrow L \underset{\text{isom.}}{\cong}$ zero section in $\mathcal{F}$.

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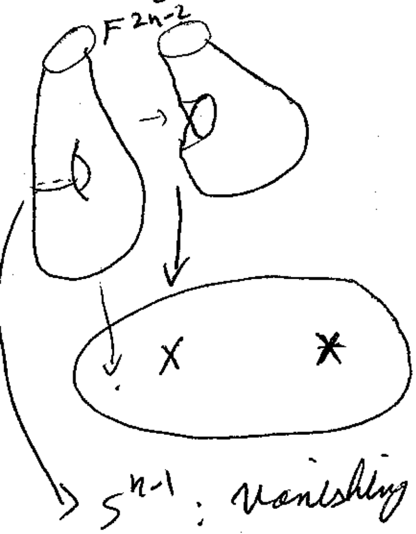
Lefschetz Fibrations: $f: M \to \underset{D^2}{\mathbb{C}}$ w/ isolated non-deg. critical points, looking like

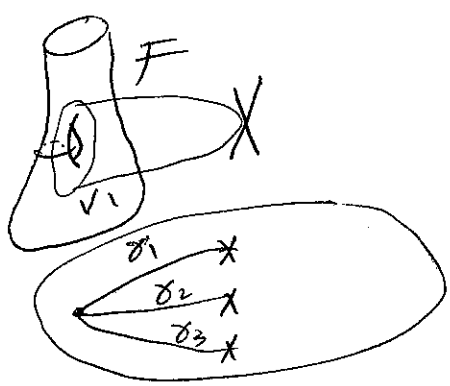
fibers are sym. sub-mflds $(z_1, \dots, z_n) \longrightarrow \sum z_i^2$

$\mathbb{C}^n$ in "good" coordinate



S^{n-1}: vanishing cycle $\longrightarrow$ D^n thimbles

Lagrangian



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$\to V_1, \dots, V_r \subset F$ vanishing cycles

$\to D_1, \dots, D_r \subset M$ thimbles

$\partial D_i = V_i$

$\mathcal{F}(\cancel{f: M \to D^2})$

$\mathcal{F}(f: M \longrightarrow D^2)$

objects: $\rightarrow$ closed exact Lag. in M

$\rightarrow$ thimbles for all paths

morphisms: Floer theory i.e. intersections

$\rightarrow$ as usual for closed Lagrangians

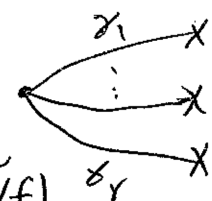
$\rightarrow$ for thimbles

$$\hom_{\mathcal{F}(f)}(D_1, D_2) := \hom_{\mathcal{F}(F)}(V_1, V_2)$$

$$\hom_{\mathcal{F}(f)}(D_1, D_1) = K\langle p\rangle$$

$$\hom_{\mathcal{F}(f)}(D_2, D_1) = 0$$

Thm: Given a basis of paths



the thimbles $D_1, \ldots, D_r$ generate $\mathcal{F}(f)$.